Geometric Boundary Data for the Gravitational Field

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An outstanding issue in the treatment of boundaries in general relativity is the lack of a local geometric interpretation of the necessary boundary data. For the Cauchy problem, the initial data is supplied by the 3-metric and extrinsic curvature of the initial Cauchy hypersurface.. This Cauchy data determines a solution to Einstein's equations which is unique up to a diffeomorphism. Here, we show how three pieces of boundary data, which are associated locally with the geometry of the boundary, likewise determine a solution of the initial-boundary value problem which is unique up to a diffeomorphism. One piece of this data, constructed from the extrinsic curvature of the boundary, determines the dynamical evolution of the boundary. The other two pieces constitute a conformal class of rank-2, positive definite metrics, which represent the two gravitational degrees of freedom.

PACS numbers: PACS number(s): 04.20.-q, 04.20.Cv, 04.20.Ex, 04.25.D-

I. INTRODUCTION

There exists a well-posed Cauchy problem for Einstein's equation which has the important property of *geometric uniqueness*, i.e. local geometric data representing the 3-metric and extrinsic curvature of the initial Cauchy hypersurface determines a spacetime metric metric g_{ab} which is unique up to diffeomorphism. There are two formulations of the initial-boundary value problem (IBVP) which are strongly well-posed, the Friedrich-Nagy formulation [1] and the harmonic formulation [2, 3], but neither give rise to a such a local version of geometric uniqueness. For the harmonic formulation, there exists a nonlocal (in time) version in which the geometric interpretation of the boundary data depends upon a background metric constructed from the initial Cauchy data [4]. As a result, the boundary data have a geometric interpretation which is nonlocal in time. In this work, we show how boundary data for the gravitational field can be posed which are locally determined by the geometry of the boundary, in the same sense as the Cauchy data.

In a Cauchy problem, initial data on a spacelike hypersurface $\mathcal{S}_0$ determine a solution in the domain of dependence $\mathcal{D}(\mathcal{S}_0)$ (which consists of those points whose past directed characteristics all intersect $\mathcal{S}_0$). In the IBVP, data on a timelike boundary $\mathcal{T}$ transverse to $\mathcal{S}_0$ are used to further extend the solution to the domain of dependence $\mathcal{D}(\mathcal{S}_0 \cup \mathcal{T})$. Strong well-posedness [5] guarantees the existence of a unique solution which depends continuously on both the initial data and the boundary data.

The primary application of the gravitational IBVP is the simulation of an isolated astrophysical system containing neutron stars and black holes. The standard approach in numerical relativity, as in computational studies of other hyperbolic systems, is to introduce an artificial outer boundary $\mathcal{T}$, which is coincident with the boundary of the computational grid and whose cross-sections are spheres surrounding the system. The ability to compute the details of the gravitational radiation produced by compact astrophysical sources, such as coalescing black holes, is of major importance to the success of gravitational wave astronomy. If the simulation of such systems is not based upon a strongly well-posed IBVP then the results cannot be trusted in the domain of dependence of the outer boundary. For comprehensive reviews of the gravitational IBVP see [6, 7].

Geometric uniqueness has the practical application of allowing numerical simulations with the same initial data but carried out with different formulations and different gauge conditions to produce geometrically equivalent (diffeomorphic) spacetimes. For a recent discussion of this issue, see [8]. For hyperbolic systems which are stable under lower order perturbations, the global solution in the spacetime manifold $\mathcal{M}$ can be obtained by patching together local solutions, i.e. the problem can be *localized*. Thus, for purposes of treating the underlying geometrical nature of the boundary data, it suffices to concentrate on the local problem in the neighborhood of a point on the boundary. That is the approach taken in this paper.

In the Friedrich-Nagy formulation, there are three essential pieces of boundary data that have geometrical or physical significance. One is the trace K of the extrinsic curvature K_{ab} of the boundary, which geometrically determines the location of the boundary. (Note that the coordinate specification of the location of the boundary is pure gauge information and does not determine its location geometrically in the same sense that a curve is determined geometrically by its acceleration, given its initial position and velocity.) Two other pieces of data in the Friedrich-Nagy

formulation, which are related to the gravitational radiation degrees of freedom, are supplied by a combination of the Weyl tensor components Ψ_0 and Ψ_4, in the Newman-Penrose notation [9]. The remaining boundary data specify the gauge freedom.

The Friedrich-Nagy formulation is based upon a symmetric hyperbolic Einstein-Bianchi system, with evolution variables consisting of an orthonormal tetrad, the associated connection coefficients and the Weyl curvature components. Although it differs from the metric based formulations used in numerical relativity, the requirement of three pieces of geometric boundary data should be universally applicable. (Statements found in the literature that only two pieces of boundary data suffice to specify the physical or geometrical properties of the gravitational field are misleading. They are only true when the boundary has been geometrically specified, e.g. for an $r =$ const boundary in a background Schwarzschild geometry.)

The outgoing null vector K^a and ingoing null vector L^a used in defining Ψ_0 and Ψ_4, respectively, are determined by the unit normal N^a to the boundary and a choice of unit timelike vector T^a tangent to the boundary according to

$$K^a = T^a + N^a , \quad L^a = T^a - N^a. \tag{1.1}$$

The choice of T^a represents gauge freedom in this data. Friedrich and Nagy are careful to point out that this gauge freedom prevents interpreting Ψ_0 and Ψ_4 as purely geometric data.

This shortcoming could perhaps be avoided by choosing these null vectors to be principle null directions of the Weyl tensor. However, in a general spacetime this would lead to four possible choices which would then have to be incorporated (in some yet unkown way) into a well-posed problem. An alternative, proposed in [8], is to base the data on the eigenvectors V^a determined by the trace-free part of the extrinsic curvature according to

$$(K_{ab} - \frac{1}{3}H_{ab}K)V^b = \lambda H_{ab}V^b, \tag{1.2}$$

where H_{ab} is the intrinsic 3-metric of the boundary. For a spherical worldtube in Minkowski space, this picks out a locally preferred timelike direction $\tilde{T}^a$. This suggests that the approach might extend to a suitably round outer boundary of an isolated system. However, it is again not clear whether such an approach can be properly incorporated into the evolution system.

Here we consider geometric boundary data for metric based formulations of the IBVP. Our main result is that, along with the initial Cauchy data, a geometrically unique spacetime is determined by three pieces of boundary data related locally to the intrinsic metric H_{ab} and extrinsic curvature K_{ab} of the boundary. More specifically, the boundary data consist of a conformal class of rank-2, positive definite metrics, which represent the two gravitational degrees of freedom, and an associated component of the extrinsic curvature, which determines the dynamical evolution of the boundary. The rank-2 metric on the 3-dimensional boundary picks out a timelike eigen-direction, with null eigenvalue, which, up to normalization, provides a geometric choice of unit timelike vector T^a tangent to the boundary. In Sec. II, we discuss the underlying geometry and present our main result as a *Local Geometric Data Theorem*.

The demonstration that this data lead to a strongly well-posed IBVP is carried out using the harmonic reduction of Einstein's equations to ten wave equations, as was the method used in establishing the analogous result for the Cauchy problem [10]. In doing so, the three pieces of local geometric boundary data must be supplemented by seven additional boundary conditions. Four of these conditions are supplied by the harmonic coordinate conditions. The other three fix the remaining freedom in the choice of harmonic coordinates. In Sec. III, the resulting harmonic IBVP is reduced to a set of partial differential equations in the frozen coefficient formalism, which are subject to a combination of Dirichlet and Neumann boundary conditions.

The demonstration that the strong well-posedness of the frozen coefficient version of the harmonic IBVP extends to the full quasilinear problem was given in [2, 3] for the case of Sommerfeld boundary conditions. In Sec. IV, we demonstrate how this result can be extended to Dirichlet and Neumann conditions.

The strong well-posedness of the frozen coefficient harmonic IBVP with local geometric boundary data is then treated in Sec. V. The key idea is that the full set of boundary conditions can be applied sequentially, similar to the approach followed in [2, 3] except now applied to a set of Dirichlet and Neumann conditions rather than Sommerfeld conditions. Our main result is then established in Sec. V D.

When the emphasis is on geometric issues we use abstract tensor indices, e.g. v^a to denote a vector field, and when the specific spacetime coordinates $x^\mu = (t, x^i)$ are introduced we use the corresponding coordinate indices, e.g. $v^\mu = (v^t, v^i)$.

II. THE INITIAL-BOUNDARY DATA

We begin with a review of the initial data for the Cauchy problem. The standard treatment of the Cauchy problem introduces a time foliation $\mathcal{S}_t$, with future directed unit normal n_a. The embedding of $\mathcal{S}_t$ in the spacetime manifold

$\mathcal{M}$ then gives rise to the decomposition of the spacetime metric

$$g_{ab} = -n_a n_b + h_{ab}, \tag{2.1}$$

where h_{ab} is the 3-metric intrinsic to $\mathcal{S}_t$. Geometric initial data are determined by the intrinsic metric h_{ab} and extrinsic curvature $k_{ab} = h_a^c \nabla_c n_b$ of the initial Cauchy hypersurface $\mathcal{S}_0$, where ∇_a is the covariant derivative associated with g_{ab}. These data are subject to the Hamiltonian and momentum constraints

$$2G^{ab} n_a n_b = \mathcal{R} - k_{ab}k^{ab} + k^2 \ = \ 0, \tag{2.2}$$

$$h_c^b G^{ac} n_a = D_b \left(k^{ab} - h^{ab} k \right) \ = \ 0, \tag{2.3}$$

where $\mathcal{R}$ is the curvature scalar and D_b is the covariant derivative associated with h_{ab}.

The remaining initial data necessary to determine a unique spacetime metric consist of gauge information, i.e. data that affect the solution only by a diffeomorphism. In the $3+1$ formulation of Einstein's equations, the gauge freedom in the metric is governed by the choice of an evolution field

$$t^a = \alpha n^a + \beta^a, \quad \beta^a n_a = 0, \tag{2.4}$$

with lapse α and shift β^a. The lapse relates the unit future-directed normal to the time foliation, according to

$$n_a = -\alpha \nabla_a t. \tag{2.5}$$

The evolution field is transverse but not in general normal to the Cauchy hypersurfaces so that it determines the shift according to

$$\beta^a = h_b^a t^b. \tag{2.6}$$

The initial data required for the formulation of a well-posed Cauchy problem depends upon the choice of hyperbolic reduction of Einstein's equations. Here we consider the hyperbolic reduction associated with harmonic coordinates, as used in the classic work of Choquet-Bruhat [10]. Generalized harmonic coordinates $x^\mu = (t, x^i) = (t, x, y, z)$ are functionally independent solutions of the curved space scalar wave equation

$$g^{ab} \nabla_a \nabla_b x^\mu = -\hat{\Gamma}^\mu, \tag{2.7}$$

where $\hat{\Gamma}^\mu(g, x)$ are harmonic gauge source functions [11]. Thus the harmonic coordinates are determined by initial data $x^\mu = (0, x^i)$ on $\mathcal{S}_0$, subject to the initial conditions $\partial_t x^\mu = \delta_t^\mu$.

In terms of the connection coefficients $\Gamma^\mu_{\alpha\beta}$, the harmonic coordinate conditions are

$$\mathcal{C}^\mu := \Gamma^\mu - \hat{\Gamma}^\mu = 0, \tag{2.8}$$

where

$$\Gamma^\mu = g^{\rho\sigma} \Gamma^\mu_{\rho\sigma} = -\frac{1}{\sqrt{-g}} \partial_\rho (\sqrt{-g} g^{\rho\mu}), \quad g = \det g_{\mu\nu}. \tag{2.9}$$

The hyperbolic reduction of the Einstein tensor results from setting

$$E^{\mu\nu} := G^{\mu\nu} - \nabla^{(\mu} \mathcal{C}^{\nu)} + \frac{1}{2} g^{\mu\nu} \nabla_\rho \mathcal{C}^\rho = 0, \tag{2.10}$$

where C^ν is treated formally as a vector field in constructing the "covariant" derivatives $\nabla^\mu C^\nu$.

When the harmonic conditions (2.8) are satisfied, the principle part of (2.10) reduces to a curved space wave operator acting on the densitized metric, i.e.

$$E^{\mu\nu} = \frac{1}{2\sqrt{-g}} g^{\alpha\beta} \partial_\alpha \partial_\beta (\sqrt{-g} g^{\mu\nu}) + \text{lower order terms} = 0. \tag{2.11}$$

Thus the harmonic evolution equations (2.10) are quasilinear wave equations for the components of the densitized metric $\sqrt{-g} g^{\mu\nu}$. The well-posedness of the Cauchy problem for the harmonic system (2.10) follows from known results for systems of quasilinear wave equations [10]. Such results are local in time since there is no general theory for the global existence of solutions to nonlinear equations.

Constraint preservation results from applying the contracted Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ to (2.10), which leads to the homogeneous wave equation

$$\nabla^\rho \nabla_\rho \mathcal{C}^\mu + R^\mu_\rho \mathcal{C}^\rho = 0. \tag{2.12}$$

If the initial data enforce

$$\mathcal{C}^\mu|_{\mathcal{S}_0} = 0 \tag{2.13}$$

and

$$\partial_t \mathcal{C}^\mu|_{\mathcal{S}_0} = 0 \tag{2.14}$$

then $\mathcal{C}^\rho = 0$ is the unique solution of (2.12). It is straightforward to satisfy (2.13) by algebraically determining the initial values of $\partial_t g^{\mu t}$ in terms of the initial values of $g^{\mu\nu}$ and their spatial derivatives. In order to see how to satisfy (2.14) note that the reduced equations (2.10) imply

$$G^{\mu\nu} n_\nu = n_\nu \nabla^{(\mu} \mathcal{C}^{\nu)} - \frac{1}{2} n^\mu \nabla_\rho \mathcal{C}^\rho. \tag{2.15}$$

As a result, if

$$G^{\mu\nu} n_\nu|_{\mathcal{S}_0} = 0, \tag{2.16}$$

i.e. if the Hamiltonian and momentum constraints are satisfied by the initial data, and if the reduced equations (2.10) are satisfied, then

$$[n_\nu \nabla^{(\mu} \mathcal{C}^{\nu)} - \frac{1}{2} n^\mu \nabla_\rho \mathcal{C}^\rho]|_{\mathcal{S}_0} = 0. \tag{2.17}$$

It is easy to check that if $\mathcal{C}^\mu|_{\mathcal{S}_0} = 0$ then (2.17) implies (2.14).

By standard results, the Hamiltonian and momentum constraints on the initial data, along with the reduced evolution equations (2.10), imply that the initial conditions (2.13) and (2.14) required for preserving the harmonic conditions are satisfied. Conversely, if the Hamiltonian and momentum constraints are satisfied initially then (2.15) ensures that they will be preserved under harmonic evolution. Thus the conditions $\mathcal{C}^\mu = 0$ substitute for the constraints of the generalized harmonic formulation.

These results show that the free initial gauge data in the harmonic formulation consist of the initial values of the lapse and shift. For simplicity, we set the initial lapse to unity and the initial shift to zero so that the metric components satisfy

$$g^{tt} = -1, \quad g^{ti} = 0, \quad t = 0. \tag{2.18}$$

Along with the initial geometric data h^{ij} and k^{ij}, these determine a unique solution to the Cauchy problem in the harmonic gauge. In geometric terms, h_{ab} and k_{ab} determine a solution which is unique up to a diffeomorphism.

We now formulate the additional geometric data necessary for the IBVP. In the IBVP, there is another natural decomposition of the metric at the boundary $\mathcal{T}$,

$$g_{ab} = N_a N_b + H_{ab}, \tag{2.19}$$

where N_a is the unit outward normal and H_{ab} is the 3-metric intrinsic to $\mathcal{T}$. The boundary $\mathcal{T}$ intersects the initial Cauchy hypersurface $\mathcal{S}_0$ at an edge $\mathcal{B}_0$. In general, the spacelike normal N_a to $\mathcal{T}$ is not orthogonal to the timelike normal n_a to $\mathcal{S}_0$. The geometric initial data must now also include the hyperbolic angle Θ_0 at the edge given by

$$\sinh \Theta_0 = N_a n^a|_{\mathcal{B}_0}. \tag{2.20}$$

The initial velocity of the boundary with respect to the initial Cauchy hypersurface is determined by Θ_0.

We represent the local geometric data on the boundary $\mathcal{T}$ which encodes the two gravitational degrees of freedom by a conformal class $\{Q_{ab}\}$ of rank 2 positive definite, symmetric tensor fields, i.e. the class defined by the equivalence relation $Q_{ab} \equiv \Omega^2 Q_{ab}$, $\Omega > 0$. Here $\{Q_{ab}\}$ is identified with the intrinsic metric of the boundary by further decomposing (2.19) into

$$H_{ab} = -T_a T_b + Q_{ab}, \tag{2.21}$$

where Q_{ab} belongs to $\{Q_{ab}\}$ and $Q_{ab}T^b = 0$. Here T^a is the future directed unit timelike vector tangent to the boundary which is geometrically picked out as an eigen-vector of $\{Q_{ab}\}$ with with null eigenvalue. On the initial Cauchy hypersurface, we identify Q_{ab} with the intrinsic metric of the edge $\mathcal{B}_0$.

The remaining part of the local geometric data on the boundary $\mathcal{T}$ is obtained from its extrinsic curvature

$$K_{ab} = H_a^c \nabla_c N_b. \tag{2.22}$$

In the Friedrich-Nagy formulation of the IBVP, the trace $K = H^{ab}K_{ab}$ forms part of the boundary data. Using the fact that H_{ab} has $(-++)$ signature, Friedrich and Nagy show that when K is expressed in terms of a boundary defining function it gives rise to a wave equation for that function which geometrically determines the location of the boundary. For the methods we use here to establish the strong well-posedness of a metric formulation of the IBVP, there does not appear to be a way to incorporate K into the boundary data. However, the alternative component

$$L = (H^{ab} - T^aT^b)K_{ab}, \tag{2.23}$$

which is geometrically determined by Q_{ab}, does supply the data in the necessary form. Because $(H^{ab} - T^aT^b)$ also has $(-++)$ signature, L geometrically determines the location of the boundary by the same construction used by Friedrich and Nagy.

We can know state our main result.

Local Geometric Data Theorem:. *Cauchy data h_{ab} and k_{ab} on $\mathcal{S}_0$ along with edge data Θ_0 on $\mathcal{B}_0$ and boundary data $\{Q_{ab}\}$ and L on $\mathcal{T}$ determine a solution of the vacuum Einstein equations (locally in time) which is unique up to a diffeomorphism. All data are assumed to be smooth and compatible.*

Here the Cauchy data must satisfy the Hamiltonian and momentum constraints but the boundary data are constraint free subject to compatibility with the Cauchy data, e.g. the restriction of $\{Q_{ab}\}$ and h_{ab} to $\mathcal{B}_0$ must lead to conformally equivalent 2-metrics. Together $\{Q_{ab}\}$ and L supply three pieces of local geometric boundary data.

As for the case of the Cauchy problem, additional data, which control the gauge degrees of freedom, are necessary to determine a unique solution. This gauge data depend upon the particular hyperbolic reduction used to formulate the IBVP. In the formulation of a strongly well-posed harmonic IBVP, the Einstein equations reduce to 10 wave equations for the components of the metric, so that 10 boundary conditions are necessary. In addition to the 3 pieces of geometric data $\{Q_{ab}\}$ and L, the harmonic conditions (2.8) supply 4 boundary conditions, as described in Sec. III. Thus 3 more pieces of gauge data on the boundary are necessary to specify completely the harmonic coordinate freedom. This data pins down the values of the harmonic coordinates on the boundary.

A non-zero value of the hyperbolic angle Θ_0 presents a technical complication in prescribing these 3 pieces of harmonic gauge data. However, the value of Θ_0 can be adjusted to zero by carrying out a Cauchy evolution in the neighborhood of $\mathcal{B}_0$ to a new choice of $\mathcal{S}_0$, which keeps $\mathcal{B}_0$ unchanged. Since the Cauchy problem is well-posed, the initial data for this modified problem depend continuously on the initial data for the original problem. Consequently, the original IBVP is strongly well-posed if the IBVP for the modified problem with $\Theta_0 = 0$ is strongly well-posed. In the following, we assume that this has been carried out. (Otherwise, the technical details in constructing a convenient gauge for establishing a well-posed IBVP become more complicated; cf. [3] where the case $\Theta_0 \neq 0$ is treated.) The requirement that $\Theta_0 = 0$ implies

$$N_a n^a|_{\mathcal{B}_0} = 0 \quad \text{so that } T^a|_{\mathcal{B}_0} = n^a|_{\mathcal{B}_0}. \tag{2.24}$$

Since harmonic coordinates are solutions of the curved space scalar wave equation, they are determined by the initial data and boundary data for a scalar wave. The boundary data for these coordinates can be specified in any form which leads to a strongly well-posed IBVP. For our present purpose, we consider Dirichlet or Neumann boundary data. In order to investigate the possible choices, let $\hat{x}^\mu = (\hat{t}, \hat{x}, \hat{x}^A)$ be Gaussian normal coordinates in the neighborhood of the boundary, where $\hat{x}^A = (\hat{y}, \hat{z})$, with the boundary at $\hat{x} = 0$ for $\hat{t} \geq 0$. In these coordinates, the metric has the form

$$g_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} dx^{\hat{\nu}} = d\hat{x}^2 + H_{\hat{I}\hat{J}} dx^{\hat{I}} dx^{\hat{J}}, \quad x^{\hat{I}} = (\hat{t}, \hat{x}^A). \tag{2.25}$$

In addition, we choose $T^a \partial_a \hat{x}^A = 0$ on the boundary so that $T^{\hat{A}} = 0$, i.e. T^a has vanishing shift relative to the $\hat{t}$-foliation of the boundary. Then the intrinsic 3-metric of the boundary has components

$$H_{\hat{I}\hat{J}} dx^{\hat{I}} dx^{\hat{J}} = T_{\hat{t}} T_{\hat{t}} d\hat{t}^2 + Q_{\hat{A}\hat{B}} dx^{\hat{A}} dx^{\hat{B}}. \tag{2.26}$$

(Note that the lapse intrinsic to the boundary could also be set to 1 locally, but globally in time such Gaussian coordinates for the intrinsic boundary metric could introduce focusing singularities.)

Harmonic coordinates $x^\mu = (t, x, x^A)$, $x^A = (y, z)$, can now be introduced by solving an IBVP for the scalar wave equation (2.7). On the boundary we prescribe the Dirichlet data $x = \hat{x} = 0$, so that the boundary is given by $x = 0$. For the remaining harmonic coordinates we prescribe the Neumann data

$$\frac{\partial x^A}{\partial \hat{x}} = 0, \quad \frac{\partial t}{\partial \hat{x}} = 0, \quad x = 0,$$

so that on the boundary

$$g^{xA} = \frac{\partial x}{\partial \hat{x}} \frac{\partial x^A}{\partial \hat{x}^\alpha} g^{\hat{x}\hat{\alpha}} = \frac{\partial x}{\partial \hat{x}} \frac{\partial x^A}{\partial \hat{x}} g^{\hat{x}\hat{x}} = 0.$$

Similarly $g^{xt} = 0$ on the boundary, which is consistent with the initial condition (2.24) at the edge $\mathcal{B}_0$. In summary, the boundary freedom in the choice of harmonic coordinates allows us to set

$$g^{xt}|_{\mathcal{T}} = g^{xA}|_{\mathcal{T}} = 0, \quad x|_{\mathcal{T}} = 0. \tag{2.27}$$

III. REDUCTION TO PDES

In order to reduce the IBVP to a set of partial differential equations (PDEs) with the initial-boundary data described in Sec. II, we express the harmonic Einstein equations (2.11) in the form

$$g^{\alpha\beta}\partial_\alpha\partial_\beta(\sqrt{-g}g^{\mu\nu}) = F^{\mu\nu}, \tag{3.1}$$

where the forcing $F^{\mu\nu}$ represents lower order terms which do not enter the principal part. Since the harmonic gauge source functions play no essential role, we set $\hat{\Gamma}^\mu = 0$.

In the harmonic coordinates defined in Sec. II, the initial data at $t = 0$, with the gauge conditions (2.18), consist of

$$\begin{aligned} g^{ij} &= h^{ij}, \quad g^{ti} = 0, \quad g^{tt} = -1, \\ \partial_t g^{ij} &= -\frac{1}{2}k^{ij}, \quad \partial_t(\sqrt{-g}g^{ti}) = -\partial_j(\sqrt{h}h^{ij}), \quad \partial_t(\sqrt{-g}g^{tt}) = 0. \end{aligned} \tag{3.2}$$

The boundary data at $x = 0$ consist of the geometric data

$$\tilde{Q}^{ab}, \quad L = (Q^{ab} - 2T^aT^b)K_{ab} = -\frac{1}{2}\sqrt{g^{xx}}(Q^{ab} - 2T^aT^b)\partial_x g_{ab}, \tag{3.3}$$

for some representative $\tilde{Q}^{ab} = \Omega^{-2}Q^{ab}$ of the conformal class $\{Q^{ab}\}$, along with the boundary gauge data (2.27),

$$g^{xt} = 0, \quad g^{xA} = 0. \tag{3.4}$$

In addition we enforce the harmonic constraint on the boundary,

$$\partial_\mu(\sqrt{-g}g^{\mu\nu})|_{x=0} = 0. \tag{3.5}$$

We now formulate the PDEs for the frozen coefficient version of the problem. The material in Sec's. IV and Sec. V shows that the strong well-posedness of this frozen coefficient problem extends to the quasilinear problem.

In this approach, the problem is localized in the neighborhood of a point p on the boundary and the wave operator in (3.1) is frozen to its value at p,

$$g^{\alpha\beta}|_{x_p}\partial_\alpha\partial_\beta. \tag{3.6}$$

By a constant linear transformation of the harmonic coordinates which keeps the x-direction fixed, we can then set $g^{\alpha\beta}|_{x_p} = \eta^{\alpha\beta}$ (the Minkowski metric). In doing so, the x-direction remains aligned with N^a and we can further align the t-direction with T^a. Then in the neighborhood of p we linearize the equations about the Minkowski metric. In terms of these coordinates $x^\mu = (t, x, x^A) = (t, x, y, z)$ and the linearized variable

$$\gamma^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}, \tag{3.7}$$

the system (3.1) takes the frozen coeffcient form

$$(-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2) \begin{pmatrix} \gamma^{tt} & \gamma^{tx} & \gamma^{ty} & \gamma^{tz} \\ \gamma^{tx} & \gamma^{xx} & \gamma^{xy} & \gamma^{xz} \\ \gamma^{ty} & \gamma^{xy} & \gamma^{yy} & \gamma^{yz} \\ \gamma^{tz} & \gamma^{xz} & \gamma^{yz} & \gamma^{zz} \end{pmatrix} = F, \quad x \geq 0, \ t \geq 0, \tag{3.8}$$

with forcing matrix F.

In these coordinates, $Q^{t\mu}=0$ and the conformal metric data consist of

$$\tilde{Q}^{AB}=Q^{1/2}Q^{AB}, \quad Q=\det Q_{AB}. \tag{3.9}$$

In the linearized approximation, this reduces to

$$\tilde{Q}^{AB}-\eta^{AB}=\gamma^{AB}-\frac{1}{2}\eta^{AB}\eta_{CD}\gamma^{CD} \tag{3.10}$$

and the extrinsic curvature data reduces to

$$L=(Q^{ab}-2T^aT^b)K_{ab}=-\frac{1}{2}\partial_x(2\gamma^{xx}+\gamma^{yy}+\gamma^{zz}). \tag{3.11}$$

Here it is assumed that this boundary data is sufficiently close to that for a boundary with vanishing extrinsic curvature in a flat spacetime to allow the iterative construction of a solution to the quasilinear problem. However, this can be arranged by considering the conformally transformed metric $g'_{\mu\nu}=\lambda^2 g_{\mu\nu}$, where $\lambda << 1$ is a positive constant; cf. p. 262 of [12]. Then $L'=\lambda L$ and, in the stretched coordinates $x'^\mu=x^\mu_p+\lambda^{-1}(x^\mu-x^\mu_p)$, the transformed metric has components $g'_{\mu'\nu'}(x')=g_{\mu\nu}(x)=\eta_{\mu\nu}+O(\lambda)$.

The boundary conditions for this linearized system are

$$\frac{1}{2}(\gamma^{yy}-\gamma^{zz})=q_1(t,y,z), \tag{3.12}$$

$$\gamma^{yz}=q_2(t,y,z), \tag{3.13}$$

$$\partial_x\left(\gamma^{xx}+\frac{1}{2}(\gamma^{yy}+\gamma^{zz})\right)=q_3(t,y,z), \tag{3.14}$$

$$\gamma^{xt}=0, \tag{3.15}$$

$$\gamma^{xy}=0, \tag{3.16}$$

$$\gamma^{xz}=0, \tag{3.17}$$

$$\partial_t\gamma^{tx}+\partial_x\gamma^{xx}+\partial_y\gamma^{xy}+\partial_z\gamma^{xz}=0, \tag{3.18}$$

$$\partial_t\gamma^{ty}+\partial_x\gamma^{xy}+\partial_y\gamma^{yy}+\partial_z\gamma^{yz}=0, \tag{3.19}$$

$$\partial_t\gamma^{tz}+\partial_x\gamma^{xz}+\partial_y\gamma^{yz}+\partial_z\gamma^{zz}=0, \tag{3.20}$$

$$\partial_t\gamma^{tt}+\partial_x\gamma^{tx}+\partial_y\gamma^{ty}+\partial_z\gamma^{tz}=0. \tag{3.21}$$

The Dirichlet data q_1 and q_2 are determined from the two conformally invariant degrees of freedom $\tilde{Q}^{AB}$. The Neumann data q_3 are determined from the extrinsic curvature scalar component L. The Dirichlet conditions (3.15) – (3.17) arise from the boundary conditions (3.4) on the harmonic coordinates. The boundary conditions (3.18) – (3.21) arise from the harmonic constraints (3.5).

IV. ENERGY ESTIMATES FOR QUASILINEAR WAVE PROBLEMS WITH SOMMERFELD, DIRICHLET OR NEUMANN BOUNDARY CONDITIONS

We establish the strong well-posedness of quasilinear IBVP for wave equations with Dirichlet and Neumann boundary conditions by an approach similar to that carried out in [3] for Sommerfeld boundary conditions. We begin by reviewing the Sommerfeld case.

A. Sommerfeld boundary conditions

The energy estimates in Sections 1–4 of [3] established that the the frozen coefficient version of the harmonic IBVP with Sommerfeld boundary conditions is unique and depends continuously on the data. In Appendix A of [3], this result was extended to the strong well-posedness of the quasilinear problem. Here we first consider a slightly simplified version of the problem treated in [3]. We show that local existence theorems and energy estimates for second order quasilinear wave equations can be obtained in the same way as for first order symmetric hyperbolic systems. It all depends on *a priori* estimates for arbitrarily high derivatives of the solutions of linear equations with variable coefficients.

Consider the half-plane problem

$$u_{tt} = Pu + Ru + F, \quad x \geq 0, \quad -\infty < y < \infty, \tag{4.1}$$

with Sommerfeld-type boundary conditions at $x = 0$,

$$\alpha(u_t + \gamma u) = u_x + q\,, \quad \alpha > 0\,, \quad \gamma > 0 \text{ strictly positive constants,} \tag{4.2}$$

smooth boundary data $q(t, y)$ and smooth initial data

$$u(0, x, y) = f_1(x, y), \quad u_t(0, x, y) = f_2(x, y). \tag{4.3}$$

Here the subscripts (t, x, y, z) denote partial derivatives, e.g $u_t = \partial_t u$,

$$Pu = (au_x)_x + (bu_y)_y - 2\gamma u_t - \gamma^2 u$$

and

$$Ru = c_1 u_t + c_2 u_x + c_3 u_y + c_4 u.$$

Ru are terms of lower (first and zeroth) differential order. Also, we use the notation

$$(u, v), \quad \|u\|^2 = (u, u); \quad (u, v)_B, \quad \|u\|_B^2 = (u, u)_B$$

to denote the L_2 scalar product and norm over the half-plane and boundary, respectively.

All coefficients and data are smooth real functions and $a \geq a_0 > 0$, $b \geq b_0 > 0$, where a_0, b_0 are strictly positive constants. The initial data are compatible with the boundary conditions. Here $\gamma > 0$ is a constant obtained by the change of variables $u \to e^{\gamma t} u'$ and then deleting the "prime". This introduces the term $\gamma^2 \|u\|^2$ in the energy

$$E := \|u_t\|^2 + (u_x, au_x) + (u_y, bu_y) + \gamma^2 \|u\|^2, \tag{4.4}$$

which provides an estimate of $\|u\|^2$.

Lemma. *There is an energy estimate which is stable against lower order perturbations.*

Proof: Integration by parts gives

$$\begin{aligned} \partial_t E &= \partial_t \left(\|u_t\|^2 + (u_x, au_x) + (u_y, bu_y) + \gamma^2 \|u\|^2 \right) \\ &= -4\gamma \|u_t\|^2 + 2(u_t, F) + 2(u_t, Ru) - 2(u_t, au_x)_B \\ &+ a_t \|u_x\|^2 + b_t \|u_y\|^2 \\ &\leq \text{const}(E + \|F\|^2) - 2(u_t, au_x)_B. \end{aligned} \tag{4.5}$$

Using the boundary conditions, we obtain

$$-(u_t, au_x)_B = -(u_t, a\alpha u_t)_B - (u_t, a\alpha\gamma u)_B + (u_t, aq)_B \leq -\frac{1}{2} a_0 \alpha\gamma \partial_t \|u\|_B^2 + \text{const}(\|u\|_B^2 + \|q\|_B^2).$$

Therefore (4.5) implies

$$\partial_t (E + a_0 \alpha\gamma \|u\|_B^2) \leq \text{const}(E + \|F\|^2 + \|u\|_B^2 + \|q\|_B^2). \tag{4.6}$$

This proves the lemma.

Now we can estimate the derivatives. Let $v = u_y,\ w = u_t$. Differentiation of the differential equation gives

$$\begin{aligned} v_{tt} &= Pv + Rv + R_y u + (a_y u_x)_x + (b_y v)_y + F_y, \\ w_{tt} &= Pw + Rw + R_t u + (a_t u_x)_x + (b_t v)_y + F_t. \end{aligned} \tag{4.7}$$

Here $R_y u$ and $R_t u$ are linear combinations of first derivatives of u which we have already estimated and can be considered part of the forcing.

The differential equation (4.1) tells us that

$$au_{xx} = w_t - bv_y + \text{ terms we have already estimated.}$$

Thus u_{xx} is lower order with respect to v and w and, except for lower order terms, v and w are solutions of the same differential equation as u. They obey the same boundary conditions with data $q_y(t,y)$ and $q_t(t,y)$, respectively. Therefore we can estimate all second derivatives. Repeating the process, we can estimate any number of derivatives.

We can now proceed in the same way as in [5], where we have considered first order systems to obtain existence theorems for equations with variable coefficients. We approximate the differential equation by a stable difference approximation and prove, using summation by parts, that the corresponding estimates for the divided differences hold independently of gridsize. In the limit of vanishing gridsize, we obtain the existence theorem. Since we can estimate any number of derivatives, it is well known, using Sobolev's theorem, that we can obtain similar, although local in time, estimates for quasilinear systems. By the same iterative methods as for first order symmetric hyperbolic systems it follows that strong well-posedness extends locally in time to the quasilinear case, as well as other standard results such as the principle of *finite speed of propagation.*

Remark. *There are no difficulties to extend the results to three spatial dimensions.*

B. Homogeneous Dirichlet and Neumann conditions

If we replace the Sommerfeld boundary conditions by homogeneous Dirichlet or Neumann conditions, then the estimates in Sec. IV A hold with boundary data $q = 0$.

Now we consider the half-plane problem for wave equations with inhomogeneous Dirichlet or Neumann boundary conditions. We want to show that we can transform these problems into problems with homogeneous boundary conditions by changing the forcing and the initial data. As a model problem, we consider the half-plane problem

$$\begin{aligned} u_{tt} &= (a(t,x,y)u_x)_x + \left(b(t,x,y)u_y\right)_y + F(t,x,y), \\ x &\geq 0, \quad -\infty < y < \infty, \quad t \geq 0, \end{aligned} \tag{4.8}$$

with initial conditions

$$u(0,x,y) = f_1(x,y), \quad u_t(0,x,y) = f_2(x,y), \tag{4.9}$$

and Dirichlet boundary conditions

$$u(t,0,y) = q(t,y). \tag{4.10}$$

We assume that all coefficients and data are compatible and smooth. We make a change of variable

$$\tilde{u}(t,x,y) = u(t,x,y) - \varphi(x)q(t,y). \tag{4.11}$$

Here $\varphi(x)$ is a smooth function, with $\varphi(0) = 1$, which decays exponentially. Then

$$\tilde{u}(t,0,y) = 0, \quad \text{i.e. } \tilde{u} \text{ satisfies homogeneous Dirichlet boundary conditions.} \tag{4.12}$$

By (4.11),

$$\begin{aligned} \tilde{u}_{tt} &= u_{tt} - (\varphi(x)q(t,y))_{tt}, \\ (a\tilde{u}_x)_x &= (au_x)_x - \left(a(\varphi(x)q(t,y)_x\right)_x, \\ (b\tilde{u}_y)_y &= (bu_y)_y - \left(b(\varphi(x)q(t,y)_y\right)_y. \end{aligned} \tag{4.13}$$

Finally, by (4.8), (4.12) and (4.13) we obtain the differential equation with modified forcing term

$$\tilde{u}_{tt} = (a(t,x,y)\tilde{u}_x)_x + (b(t,x,y)\tilde{u}_y)_y + F + \tilde{F}, \tag{4.14}$$

which satisfies homogenous Dirichlet boundary conditions. By assumption, F is a smooth function and $\tilde{F}$ is composed of a, b, φ and q and their first two derivatives. Since derivatives are smooth functions, $\tilde{F}$ is also a smooth function. Therefore $\tilde{u}(t,x,y)$ satisfies the estimates arrived at in Sec. IV A.

Now we consider (4.8) with the Neumann boundary condition

$$u_x(t,0,y) = q(t,y). \tag{4.15}$$

We make again the transformation (4.11) but now with $\varphi_x(0) = 1$, and obtain the corresponding result.

As an illustration of how the estimates extend to higher derivatives, consider the half-plane problem (4.8) with a homogeneous Dirichlet boundary condition for $a = b = 1$ (which poses no restriction),

$$u_{tt} = u_{xx} + u_{yy} + F, \quad u(t,0,y) = 0. \tag{4.16}$$

By differentiation, we obtain

$$u_{ytt} = u_{yxx} + u_{yyy} + F_y, \quad u_y(t,0,y) = 0, \tag{4.17}$$

$$u_{ttt} = u_{txx} + u_{tyy} + F_t, \quad u_t(t,0,y) = 0. \tag{4.18}$$

As in (4.7), we introduce the variables

$$v = u_y, \quad w = u_t. \tag{4.19}$$

Then (4.17), (4.18) become

$$v_{tt} = v_{xx} + v_{yy} + F_y, \quad v(t,0,y) = 0, \tag{4.20}$$

$$w_{tt} = w_{xx} + w_{yy} + F_t, \quad w(t,0,y) = 0. \tag{4.21}$$

Integration by parts then gives us an energy estimate for

$$\|v_t\|^2 + \|v_x\|^2 + \|v_y\|^2 + \|w_t\|^2 + \|w_x\|^2 + \|w_y\|^2. \tag{4.22}$$

By (4.16) and (4.19) we obtain

$$u_{xx} + F = u_{tt} - u_{yy} = w_t - v_y.$$

Therefore, by (4.22), we obtain a bound for $\|u_{xx}\|^2$.

We obtain a bound for $\|u_{xxx}\|^2$ in the same way by replacing v and w by

$$v^{(1)} = u_{yy}, \quad w^{(1)} = u_{tt}. \tag{4.23}$$

Now we obtain the differential equations

$$\begin{aligned} v^{(1)}_{tt} &= v^{(1)}_{xx} + v^{(1)}_{yy} + F_{yy}, \quad v^{(1)}(t,0,y) = 0, \\ w^{(1)}_{tt} &= w^{(1)}_{xx} + w^{(1)}_{yy} + F_{tt}, \quad w^{(1)}(t,0,y) = 0, \end{aligned} \tag{4.24}$$

and we obtain energy estimates for

$$\|v^{(1)}_t\|^2 + \|v^{(1)}_x\|^2 + \|v^{(1)}_y\|^2 + \|w^{(1)}_t\|^2 + \|w^{(1)}_x\|^2 + \|w^{(1)}_y\|^2 \ , \tag{4.25}$$

which we can express in terms of u according to

$$\|u_{tyy}\|^2 + \|u_{xyy}\|^2 + \|u_{yyy}\|^2 + \|u_{ttt}\|^2 + \|u_{xtt}\|^2 + \|u_{ytt}\|^2. \tag{4.26}$$

By differentiation of (4.16) with respect to x,

$$u_{xxx} = u_{xtt} - u_{xyy} - F_x. \tag{4.27}$$

From (4.26), we already have estimates for $\|u_{xtt}\|^2$ and $\|u_{xyy}\|^2$. Therefore we also obtain an estimate for $\|u_{xxx}\|^2$. This process can be continued.

Our result is not restricted to the model problem but is valid in general. For example, we can replace (4.8) by the corresponding half-plane problem in three spatial dimensions.

Remark. *In many problems, surface waves, glancing waves and other waves specific to the boundary are important. In that case, there is no energy estimate and the above technique does not activate these phenomena. Instead, in such cases, we split the problem into two problems; one with homogeneous boundary conditions and another where only the boundary conditions do not vanish, i.e. the forcing and the initial values are zero. The first is covered by Sec. IV. The second we treat by Fourier-Laplace techniques. For examples, see [13, 14].*

V. THE STRONG WELL-POSEDNESS OF THE IBVP FOR THE HARMONIC EINSTEIN EQUATIONS

Here we establish the strong well-posedness of the gravitational IBVP for the system (3.8) with boundary conditions (3.12) – (3.21) determined by local geometric boundary data and harmonic coordinated conditions. In order to illustrate how the estimates in Sec. IV apply we progress through a sequence of model problems.

A. Model problem I: The harmonic Einstein equations in one spatial dimension

First consider the half-plane problem in the frozen coefficient formalism of the harmonic Einstein equations for the system of wave equations in one space variable

$$(-\partial_t^2+\partial_x^2)\begin{pmatrix}\gamma^{tt} & \gamma^{tx}\\ \gamma^{tx} & \gamma^{xx}\end{pmatrix}=F, \quad x\ge 0,\ t\ge 0, \tag{5.1}$$

with forcing matrix F. In standard notation, we treat the system in the sequential order

$$\begin{array}{ll} (1) & \partial_t^2\gamma^{tx}=\partial_x^2\gamma^{tx}+F_1,\\ (2) & \partial_t^2\gamma^{xx}=\partial_x^2\gamma^{xx}+F_2,\\ (3) & \partial_t^2\gamma^{tt}=\partial_x^2\gamma^{tt}+F_3. \end{array} \tag{5.2}$$

Here $\gamma^{tx}(t,x),\ \gamma^{xx}(t,x),\ \gamma^{tt}(t,x)$ denote the dependent variables which we want to determine on the half-plane. The forcing terms $F_1(t,x),\ F_2(t,x),\ F_3(t,x)$ are smooth functions of (t,x).

The solution of our problem is determined by the initial data corresponding to (3.2),

$$\begin{array}{ll} \text{the Dirichlet boundary condition} & \gamma^{tx}(t,0)=q(t)\\ \text{or the Neumann boundary condition} & \partial_x\gamma^{tx}(t,0)=q(t), \end{array} \tag{5.3}$$

and the harmonic constraints, which we apply on the boundary in the sequential order

$$\partial_t\gamma^{tx}(t,0)+\partial_x\gamma^{xx}(t,0) = 0, \tag{5.4}$$

$$\partial_t\gamma^{tt}(t,0)+\partial_x\gamma^{tx}(t,0) = 0. \tag{5.5}$$

We start with the wave equation for γ^{tx} with smooth boundary data (5.3) and smooth compatible initial data. By means of the transformation (4.11) in Sec. IV B, we modify the forcing so that the variables satisfy homogeneous boundary conditions, which we denote by

$$q(t)\equiv 0. \tag{5.6}$$

Then there is an energy estimate. The problem is strongly well-posed and we can solve the wave equation to estimate γ^{tx} and its derivatives on the boundary, as well as in the interior $x>0$, in terms of the data. Next we use the constraint (5.4) and obtain Neumann boundary data for γ^{xx} from $\partial_t\gamma^{tx}(t,0)$. We again use the transformation (4.11) so that $\partial_x\gamma^{xx}(t,0)\equiv 0$, using the notation (5.6). The resulting wave problem for γ^{xx} with homogeneous Neumann data is strongly well-posed so that we can estimate $\gamma^{xx}(t,x)$ and its derivatives. Finally, we obtain the same result for γ^{tt}, using the constraint (5.5) and the transformation (4.11).

B. Model problem II: The harmonic Einstein equations in two spatial dimensions

Now consider the half-plane problem in frozen coefficient formalism in two spatial dimensions,

$$\left(-\partial_t^2+\partial_x^2+\partial_y^2\right)\begin{pmatrix}\gamma^{tt} & \gamma^{tx} & \gamma^{ty}\\ \gamma^{tx} & \gamma^{xx} & \gamma^{xy}\\ \gamma^{ty} & \gamma^{xy} & \gamma^{yy}\end{pmatrix}=F, \quad x\geq 0,\ t\geq 0,\ -\infty<y<\infty,$$

where F again represents the forcing. The components γ^{yy}, γ^{tx} and γ^{xy} satisfy Dirichlet or Neumann boundary conditions. The initial data correspond to (3.2).

The harmonic constraints are applied on the boundary in the sequential order

$$\partial_t\gamma^{tx}(t,0,y)+\partial_x\gamma^{xx}(t,0,y)+\partial_y\gamma^{xy}(t,0,y) = 0, \tag{5.7}$$

$$\partial_t\gamma^{ty}(t,0,y)+\partial_x\gamma^{xy}(t,0,y)+\partial_y\gamma^{yy}(t,0,y) = 0, \tag{5.8}$$

$$\partial_t\gamma^{tt}(t,0,y)+\partial_x\gamma^{tx}(t,0,y)+\partial_y\gamma^{ty}(t,0,y) = 0. \tag{5.9}$$

We proceed essentially in the same way as for model problem I. We use the transformation (4.11) such that wave equations for γ^{yy}, γ^{tx} and γ^{xy} satisfy homogeneous Dirichlet or Neumann boundary conditions. Then there is an energy estimate for these variables and we can use the constraints to obtain estimates for the remaining variables. The constraint (5.7) determines Neumann boundary data for $\gamma^{xx}(t,0,y)$. After using the transformation (4.11), it reduces to

$$\partial_x\gamma^{xx}(t,0,y)\equiv 0 \tag{5.10}$$

and the resulting wave problem for γ^{xx} is strongly well-posed. Thus we can estimate $\gamma^{xx}(t,x)$ and its derivatives. Similarly, the constraint (5.8) determines Dirichlet boundary data for $\gamma^{ty}(t,0,y)$. After the transformation (4.11), the constraint (5.8) reduces to

$$\partial_t\gamma^{ty}(t,0,y)\equiv 0 \tag{5.11}$$

and the resulting wave problem for γ^{ty} is strongly well-posed. The constraint (5.8) now determines Dirichlet boundary data for $\gamma^{tt}(t,0,y)$ and we can use the transformation (4.11) to obtain a a strongly well-posed problem for γ^{tt}.

C. Model problem III: The harmonic Einstein equations in three spatial dimensions

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We now consider the half-plane problem for the linearized harmonic equations in three spatial dimensions

$$\left(-\partial_t^2+\partial_x^2+\partial_y^2+\partial_z^2\right)\begin{pmatrix}\gamma^{tt} & \gamma^{tx} & \gamma^{ty} & \gamma^{tz}\\ \gamma^{tx} & \gamma^{xx} & \gamma^{xy} & \gamma^{xz}\\ \gamma^{ty} & \gamma^{xy} & \gamma^{yy} & \gamma^{yz}\\ \gamma^{tz} & \gamma^{zx} & \gamma^{zy} & \gamma^{zz}\end{pmatrix}=F, \tag{5.12}$$

$$x\geq 0,\ t\geq 0,\ -\infty<y<\infty,\ -\infty<z<\infty,$$

where F represents the forcing, γ^{yy}, γ^{yz}, γ^{zz}, γ^{tx}, γ^{xy} and γ^{xz} satisfy Dirichlet or Neumann boundary conditions and the initial data correspond to (3.2).

The harmonic constraints are applied on the boundary in the sequential order

$$\partial_t\gamma^{tx}+\partial_x\gamma^{xx}+\partial_y\gamma^{xy}+\partial_z\gamma^{xz} = 0, \tag{5.13}$$

$$\partial_t\gamma^{ty}+\partial_x\gamma^{xy}+\partial_y\gamma^{yy}+\partial_z\gamma^{yz} = 0, \tag{5.14}$$

$$\partial_t\gamma^{tz}+\partial_x\gamma^{xz}+\partial_y\gamma^{zy}+\partial_z\gamma^{zz} = 0, \tag{5.15}$$

$$\partial_t\gamma^{tt}+\partial_x\gamma^{tx}+\partial_y\gamma^{ty}+\partial_z\gamma^{tz} = 0. \tag{5.16}$$

We proceed in the same way as in two space dimensions. We use the transformation (4.11) so that the six wave equations for γ^{yy}, γ^{yz}, γ^{zz}, γ^{tx}, γ^{xy} and γ^{xz} satisfy homogeneous Dirichlet or Neumann boundary conditions. Then there is an energy estimate for these variables and their derivatives. We then use the constraints to obtain estimates for the remaining variables.

The constraints (5.13) - (5.15) determine Neumann boundary data for $\partial_x\gamma^{xx}(t,0,y,z)$ and Dirichlet boundary data for $\gamma^{ty}(t,0,y,z)$ and $\gamma^{tz}(t,0,y,z)$ in terms of the previously estimated variables.. After using the transformation (4.11), the resulting wave problem is strongly well-posed so that we can estimate γ^{xx}, γ^{ty} and γ^{tz} and their derivatives. The constraint (5.16) then provides Dirichlet data $\gamma^{tt}(t,0,y,z)$ for the remaining variable in terms of previously estimated variables After the transformation (4.11), the resulting wave problem for γ^{tt} is strongly well-posed.

D. The harmonic Einstein equations with local geometric data

Now we turn to the 3-dimensional harmonic Einstein system (5.12) with boundary conditions (3.12) – (3.21) determined by local geometric boundary data and harmonic coordinate conditions, as prescribed in Sec. III. After applying the transformation (4.11), the conformal metric data (3.12) – (3.13) reduce to the homogeneous Dirichlet form

$$(\gamma^{yy} - \gamma^{zz})(t, 0, y, z) \equiv 0, \quad \gamma^{yz}(t, 0, y, z) \equiv 0, \tag{5.17}$$

and the extrinsic curvature data L (3.14) reduce to the homogeneous Neumann form

$$\partial_x(\gamma^{yy} + \gamma^{zz} + 2\gamma^{xx})(t, 0, y, z) \equiv 0. \tag{5.18}$$

The boundary gauge data (3.15) – (3.17) are already in the homogeneous Dirichlet form

$$\gamma^{tx}(t, 0, y, z) = \gamma^{xy}(t, 0, y, z) = \gamma^{xz}(t, 0, y, z) = 0. \tag{5.19}$$

The remaining boundary conditions are supplied by the harmonic constraints (5.13) - (5.16).

The situation is similar to model problem III but simpler since the gauge conditions (5.19) are already homogeneous and imply that the constraint (5.13) reduces to the homogeneous form

$$\partial_x \gamma^{xx}(t, 0, y, z) = 0 \tag{5.20}$$

so that (5.18) reduces to

$$\partial_x(\gamma^{yy} + \gamma^{zz})(t, 0, y, z) \equiv 0. \tag{5.21}$$

Together the homogeneous boundary conditions (5.17), (5.19), (5.20) and (5.21) determine strongly well-posed wave problems for $(\gamma^{yy}-\gamma^{zz})$, γ^{yz}, γ^{tx}, γ^{xy}, γ^{xz}, γ^{xx} and $(\gamma^{yy}+\gamma^{zz})$, respectively. Thus we can estimate those variables and their derivatives. Now we can proceed as in Model problem III to use the constraints (5.14) - (5.16) in sequential order to determine the required estimates for the remaining three variables γ^{ty}, γ^{tz} and γ^{tt}. Along with the applicability to the quasilnear problem outlined in Sec. IV, this establishes the *Local Geometric Data Theorem* proposed in Sec. II.

VI. DISCUSSION

We have shown that the three pieces of locally geometric boundary data supplied by a conformal class of rank-2 metrics $\{Q_{ab}\}$ and an associated extrinsic curvature component L determine, up to gauge, the necessary boundary data for a solution of Einstein's equations. This resolves an outstanding issue in the geometrical understanding of the gravitational IBVP.

The method used to establish this result also broadens the possible formulations of a strongly well-posed harmonic IBVP. The method in [2, 3] based upon Sommerfeld conditions has been extended to include Dirichlet and Neumann conditions, subject to the sequential structure necessary to enforce the harmonic constraints. For computational applications, Sommerfeld conditions are most benevolent because they allow numerical error to leave the grid. It is therefore somewhat discordant with numerical application that a treatment of the boundary based upon locally geometric data must apparently include at least two Dirichlet conditions, associated with $\{Q_{ab}\}$, and one Neumann condition, associated with L.

There are many options in formulating a suitable combination of Dirichlet, Neumann and Sommerfeld conditions for a strongly weel-posed problem, provided the sequential structure is maintained. However, none of these options seem to be consistent with a choice of locally geometric boundary data different from $\{Q_{ab}\}$ and L. it is possible that other choices of might be established by a different approach. For example, had we used the trace K of the extrinsic curvature of the boundary instead of the component L then (5.21) would have been replaced by

$$\partial_x(\gamma^{yy} + \gamma^{zz} - \gamma^{tt})(t, 0, y, z) \equiv 0. \tag{6.1}$$

This does not fit into the sequential structure for applying the constraints but it is still possible that the resulting system is strongly well-posed. It depends upon whether the higher derivatives can be estimated by some generalization of the scalar treatment given in Sec. IV.

An additional issue of practical importance is the formulation of a strongly well-posed IBVP for the $3+1$ approach which has historically played a major role in numerical relativity [15]. In the $3+1$ formalism, instead of the 10 wave equations of the harmonic system, Einstein's equations are reduced to a pair of 6 first order in time equations for h_{ab} and k_{ab}, supplemented by 4 conditions which determine the lapse and shift. Perhaps the geometric insight provided by our results can be transferred to shed light on this problem.

Acknowledgments

We are grateful for numerous discussions with H. Friedrich, which supplied the catalyst for this work. The research was supported by NSF grants PHY-0854623 and PHY-1201276 to the University of Pittsburgh.

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